COUPLED PAINLEVÉ VI SYSTEMS IN DIMENSION FOUR WITH AFFINE WEYL GROUP SYMMETRY OF TYPE $E_6^{(2)}$

YUSUKE SASANO

ABSTRACT. We find a four-parameter family of coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type $E_6^{(2)}$. This is the first example which gave higher order Painlevé type systems of type $E_6^{(2)}$. We study its symmetry and holomorphy conditions.

1. INTRODUCTION

In [10, 11, 14, 13, 12, 15, 16], we presented some types of coupled Painlevé systems with various affine Weyl group symmetries by connecting the invariant divisors $p_i, q_i - q_{i+1}, p_{i+1}$ for the canonical variables (q_i, p_i) $(i = 1, 2, \dots, n)$. These systems are polynomial Hamiltonian systems with coupled Painlevé Hamiltonians.

In this paper, we find a 4-parameter family of coupled Painlevé VI systems in dimension four with affine Weyl group symmetry of type $E_6^{(2)}$ given by

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z} \tag{1}$$

with the polynomial Hamiltonian

$$\begin{aligned} t(1-t)H =& x^2y^3 + ((1-2t)x - 2\alpha_1 - \alpha_2 - \alpha_3)xy^2 \\ & + \{(t-1)tx^2 + ((4\alpha_1 + 4\alpha_2 + 3\alpha_3 + \alpha_4)t - (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4))x \\ & + \alpha_1(\alpha_1 + \alpha_2 + \alpha_3)\}y - (1-t)t\alpha_0 x \\ & + \frac{1}{4}[-z^2w^4 + 2\alpha_3 zw^3 + ((1+t)z^2 + 2(2\alpha_1 + 2\alpha_2 + \alpha_3)z - \alpha_3^2)w^2 \\ & - 2\{((-2\alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4)t + (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4))z \\ & + \alpha_3(2\alpha_1 + 2\alpha_2 + \alpha_3)\}w - t(z + 4\alpha_1 + 4\alpha_2 + 2\alpha_3)z] \\ & + (txz + (1-t)xzw - xzw^2 - xyz + xyzw - \alpha_1(w-1)z + \alpha_3 xw)y. \end{aligned} \tag{2}$$

Here x, y, z and w denote unknown complex variables, and $\alpha_0, \alpha_1, \dots, \alpha_4$ are complex parameters satisfying the relation:

$$\alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 = 1. \tag{3}$$

In section 2, each principal part of this Hamiltonian can be transformed into canonical Painlevé VI Hamiltonian (6) by birational and symplectic transformations.

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This is the first example which gave higher order Painlevé type systems of type $E_6^{(2)}$.

We remark that for this system we tried to seek its first integrals of polynomial type with respect to x, y, z, w. However, we can not find. Of course, the Hamiltonian H is not the first integral.

It is known that the Painlevé VI system admits the affine Weyl group symmetry of type $F_4^{(1)}$ (see [17]) as the group of its Bäcklund transformations in addition to the diagram automorphisms of type $D_4^{(1)}$. The diagram automorphisms change the time variable t. However, in section 3, the system (1) admits the affine Weyl group symmetry of type $E_6^{(2)}$ as the group of its Bäcklund transformations, whose generators $s_0, s_1, \dots, s_4$ are determined by the invariant divisors (3.2). Of course, these transformations do not change the time variable t.

2. Principal parts of the Hamiltonian

In this section, we study two Hamiltonians K_1 and K_2 in the Hamiltonian H.

At first, we study the Hamiltonian system

$$\frac{dx}{dt} = \frac{\partial K_1}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial K_1}{\partial x} \tag{4}$$

with the polynomial Hamiltonian

$$\begin{aligned} t(1-t)K_1 =& x^2y^3 + ((1-2t)x - 2\alpha_1 - \alpha_2 - \alpha_3)xy^2 \\ &+ \{(t-1)tx^2 + ((4\alpha_1 + 4\alpha_2 + 3\alpha_3 + \alpha_4)t - (2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4))x \\ &+ \alpha_1(\alpha_1 + \alpha_2 + \alpha_3)\}y - (1-t)t\alpha_0 x, \end{aligned} \tag{5}$$

where setting $z = w = 0$ in the Hamiltonian H, we obtain K_1.

We transform the Hamiltonian (5) into the Painlevé VI Hamiltonian:

$$\begin{aligned} &H_{VI}(x, y, t; \beta_0, \beta_1, \beta_2, \beta_3, \beta_4) \\ &= \frac{1}{t(t-1)}[y^2(x-t)(x-1)x - \{(\beta_0 - 1)(x-1)x + \beta_3(x-t)x \\ &+ \beta_4(x-t)(x-1)\}y + \beta_2(\beta_1 + \beta_2)x] \quad (\beta_0 + \beta_1 + 2\beta_2 + \beta_3 + \beta_4 = 1). \end{aligned} \tag{6}$$

Step 1: We make the change of variables:

$$x_1 = x, \quad y_1 = y - t. \tag{7}$$

Step 2: We make the change of variables:

$$x_2 = -y_1, \quad y_2 = x_1. \tag{8}$$

Then, we can obtain the Painlevé VI Hamiltonian:

$$H_{VI}(x_2, y_2, t; \alpha_0, \alpha_2 + \alpha_3, \alpha_1, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2). \tag{9}$$

Of course, the parameters α_i and β_j satisfy the relations:

$$\beta_0 + \beta_1 + 2\beta_2 + \beta_3 + \beta_4 = \alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 = 1. \tag{10}$$

We remark that all transformations are symplectic.

Next, we study the Hamiltonian system

$$\frac{dx}{dt}=\frac{\partial K_2}{\partial y},\quad \frac{dy}{dt}=-\frac{\partial K_2}{\partial x} \tag{11}$$

with the polynomial Hamiltonian

$$\begin{aligned} t(1-t)K_2 =&\frac{1}{4}[-z^2w^4+2\alpha_3zw^3+((1+t)z^2+2(2\alpha_1+2\alpha_2+\alpha_3)z-\alpha_3^2)w^2\\ &-2\{((-2\alpha_1-2\alpha_2-\alpha_3-\alpha_4)t+(2\alpha_1+2\alpha_2+2\alpha_3+\alpha_4))z\\ &+\alpha_3(2\alpha_1+2\alpha_2+\alpha_3)\}w-t(z+4\alpha_1+4\alpha_2+2\alpha_3)z], \end{aligned} \tag{12}$$

where setting $x=y=0$ in the Hamiltonian H, we obtain K_2.

Let us transform the Hamiltonian (12) into the Painlevé VI Hamiltonian.

Step 1: We make the change of variables:

$$t=T_1^2. \tag{13}$$

We note that

$$dK_2\wedge dt=2T_1d\tilde{K}_2\wedge dT_1. \tag{14}$$

Step 2: We make the change of variables:

$$z_1=2z,\quad w_1=\frac{1}{2}w+\frac{1}{2}. \tag{15}$$

By this transformation, in the coordinate system $(Z_1,W_1)=(1/z_1,w_1)$ two of four accessible singular points are transformed into $W_1=0$ and $W_1=1$.

Step 3: We make the change of variables:

$$z_2=-(z_1w_1-\alpha_3)w_1,\quad w_2=\frac{1}{w_1}. \tag{16}$$

Step 4: We make the change of variables:

$$z_3=\frac{T_1-1}{T_1+1}z_2,\quad w_3=\frac{T_1+1}{T_1-1}w_2+\frac{2}{1-T_1},\quad T_1=1-2T_2+2\sqrt{T_2(T_2-1)}. \tag{17}$$

By this transformation, in the coordinate system $(Z_2,W_2)=(1/z_4,w_4)$ the others are transformed into $W_2=0$ and $W_2=\frac{1}{T_2}$. We remark that it is not $W_2=\infty$ but $W_2=0$ because we consider in the coordinate system (z_2,w_2).

Step 5: We make the change of variables:

$$z_4=-(z_3w_3-\alpha_3)w_3,\quad w_4=\frac{1}{w_3}. \tag{18}$$

Step 6: We make the change of variables:

$$z_5=w_4,\quad w_5=-z_4. \tag{19}$$

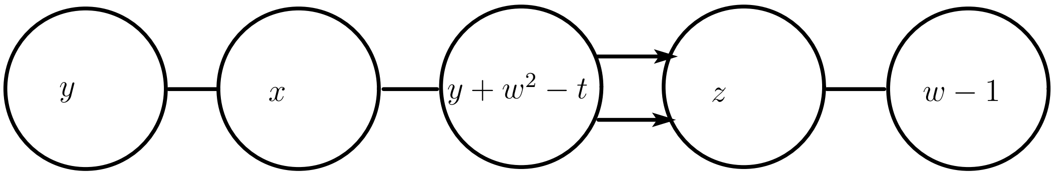

Figure 1. The figure denotes the Dynkin diagram of type $E_6^{(2)}$. The symbol in each circle denotes the invariant divisors of the system (1) of type $E_6^{(2)}$.

Then, we can obtain the Painlevé VI Hamiltonian:

$$\frac{1}{2}H_{VI}(z_5, w_5, T_2; \alpha_0+\alpha_2-1, \alpha_0+\alpha_2, \alpha_3, \alpha_4, 1-\alpha_0+2\alpha_1+\alpha_2). \tag{20}$$

We remark that all transformations are symplectic.

3. Symmetry and holomorphy conditions

In this section, we study the symmetry and holomorphy conditions of the system (1). These properties are new.

Theorem 3.1. *The system* (1) *admits the affine Weyl group symmetry of type* $E_6^{(2)}$ *as the group of its Bäcklund transformations, whose generators* $s_0, s_1, \dots, s_4$ *defined as follows*: *with the notation* $(*) := (x, y, z, w, t; \alpha_0, \alpha_1, \dots, \alpha_4)$:

(21)
$$\begin{aligned}
&s_0 : (*) \rightarrow \left(x+\frac{\alpha_0}{y}, y, z, w, t; -\alpha_0, \alpha_1+\alpha_0, \alpha_2, \alpha_3, \alpha_4\right),\\
&s_1 : (*) \rightarrow \left(x, y-\frac{\alpha_1}{x}, z, w, t; \alpha_0+\alpha_1, -\alpha_1, \alpha_2+\alpha_1, \alpha_3, \alpha_4\right),\\
&s_2 : (*) \rightarrow \left(x+\frac{\alpha_2}{y+w^2-t}, y, z+\frac{2\alpha_2 w}{y+w^2-t}, w, t; \alpha_0, \alpha_1+\alpha_2, -\alpha_2, \alpha_3+2\alpha_2, \alpha_4\right),\\
&s_3 : (*) \rightarrow \left(x, y, z, w-\frac{\alpha_3}{z}, t; \alpha_0, \alpha_1, \alpha_2+\alpha_3, -\alpha_3, \alpha_4+\alpha_3\right),\\
&s_4 : (*) \rightarrow \left(x, y, z+\frac{\alpha_4}{w-1}, w, t; \alpha_0, \alpha_1, \alpha_2, \alpha_3+\alpha_4, -\alpha_4\right).
\end{aligned}$$

We note that the Bäcklund transformations of this system satisfy

$$s_i(g) = g + \frac{\alpha_i}{f_i}\{f_i, g\} + \frac{1}{2!}\left(\frac{\alpha_i}{f_i}\right)^2 \{f_i, \{f_i, g\}\} + \cdots \quad (g \in \mathbb{C}(t)[x, y, z, w]), \tag{22}$$

where poisson bracket $\{,\}$ satisfies the relations:

$$\{y, x\} = \{w, z\} = 1, \quad \textit{the others are } 0.$$

Since these Bäcklund transformations have Lie theoretic origin, similarity reduction of a Drinfeld-Sokolov hierarchy admits such a Bäcklund symmetry.

PROPOSITION 3.2. *This system has the following invariant divisors*:

parameter's relation	f_i
$\alpha_0 = 0$	$f_0 := y$
$\alpha_1 = 0$	$f_1 := x$
$\alpha_2 = 0$	$f_2 := y + w^2 - t$
$\alpha_3 = 0$	$f_3 := z$
$\alpha_4 = 0$	$f_4 := w - 1$

We note that when $\alpha_0 = 0$, we see that the system (1) admits a particular solution $y = 0$, and when $\alpha_2 = 0$, after we make the birational and symplectic transformations:

$$x_2 = x, \ y_2 = y + w^2 - t, \ z_2 = z - 2xw, \ w_2 = w \tag{23}$$

we see that the system (1) admits a particular solution $y_2 = 0$.

THEOREM 3.3. *Let us consider a polynomial Hamiltonian system with Hamiltonian $K \in \mathbb{C}(t)[x, y, z, w]$. We assume that*

(A1) $deg(K) = 6$ with respect to x, y, z, w.

(A2) This system becomes again a polynomial Hamiltonian system in each coordinate system r_i $(i = 0, 1, \ldots, 4)$:

$$\begin{aligned}
&r_0 : x_0 = \frac{1}{x}, \ y_0 = -(yx + \alpha_0)x, \ z_0 = z, \ w_0 = w, \\
&r_1 : x_1 = -(xy - \alpha_1)y, \ y_1 = \frac{1}{y}, \ z_1 = z, \ w_1 = w, \\
&r_2 : x_2 = \frac{1}{x}, \ y_2 = -\left((y + w^2 - t)x + \alpha_2\right)x, \ z_2 = z - 2xw, \ w_2 = w, \\
&r_3 : x_3 = x, \ y_3 = y, \ z_3 = -(zw - \alpha_3)w, \ w_3 = \frac{1}{w}, \\
&r_4 : x_4 = x, \ y_4 = y, \ z_4 = \frac{1}{z}, \ w_4 = -((w - 1)z + \alpha_4)z.
\end{aligned} \tag{24}$$

Then such a system coincides with the system (1) *with the polynomial Hamiltonian* (2).

By this theorem, we can also recover the parameter's relation (3).

We note that the condition $(A2)$ should be read that

$$r_j(K) \quad (j = 0, 1, 3, 4), \quad r_2(K + x)$$

are polynomials with respect to x, y, z, w.

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